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A case study of global stability of strong rarefaction waves for 2×2 hyperbolic conservation laws with artificial viscosity

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Abstract

This paper is concerned with the global stability of strong rarefaction waves for a class of 2×2 hyperbolic conservation laws with artificial viscosity, i.e., the p-system with artificial viscosity

$$\begin{cases} v_t - u_x = \varepsilon_1 v_{xx}, \\ u_t + p(v)_x = \varepsilon_2 u_{xx}, \\ (v,u)|_{t=0} = (v_0,u_0)(x) \to (v_\pm, u_\pm) \quad \text{as } x \to \pm\infty, \end{cases}$$

where ε_i $(i = 1, 2)$ are positive constants and $p(v)$ is a smooth function defined on $v > 0$ satisfying $p'(v) < 0$, $p''(v) > 0$ for $v > 0$.

Let $(V(t,x), U(t,x))$ be the smooth approximation of the rarefaction wave profile constructed similar to that of [A. Matsumura, K. Nishihara, Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas, Comm. Math. Phys. 144 (1992) 325–335], if the H^1-norm of the initial perturbation $(v_0(x) - V(0,x),\ u_0(x) - U(0,x))$ is small, the nonlinear stability of strong rarefaction waves is well-understood, but for the case when $\|(v_0(x) - V(0,x), u_0(x) - U(0,x))\|_{H^1}$ is large, to our knowledge, fewer results have been obtained and in this paper, we obtain two types of results in this direction. Roughly speaking, if $\varepsilon_1 \neq \varepsilon_2$, we can get the nonlinear stability result provided that $\|(v_0(x) - V(0,x), u_0(x) - U(0,x))\|_{L^2}$ is small. In some sense it is a generalization of the result obtained in [D. Hoff, J.A. Smoller, Solutions in the large for certain nonlinear parabolic systems, Ann. Inst. H. Poincaré 2 (1985) 213–235] for the case $(v_-, u_-) = (v_+, u_+)$ to the case $(v_-, u_-) \neq (v_+, u_+)$ and the

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method developed by Y. Kanel' in [Y. Kanel', On a model system of equations of one-dimensional gas motion (in Russian), Differ. Uravn. 4 (1968) 374–380] plays an essential role in obtaining the uniform lower bound for $v(t,x)$. While for the case when $\varepsilon_1=\varepsilon_2$, the above system admits positively invariant regions which yields the uniform lower bound for $v(t,x)$ and based on this, two types of global stability results are obtained: first, for general flux function $p(v)$, if $\|(v_0(x)-V(0,x),u_0(x)-U(0,x))\|_{H^1}$ depends on t_0, a sufficiently large positive constant introduced in constructing the smooth approximation to the rarefaction wave solution, some restrictions on its growth rate as $t_0\to+\infty$ must be imposed. While for some special flux functions $p(v)$ which contain $p(v)=v^{-\gamma}$ ($\gamma\geqslant 1$) as a special case, similar result holds for any $(v_0(x)-V(0,x),u_0(x)-U(0,x))\in H^1(\mathbf{R})$.

MSC: 35Q40; 35L60

Keywords: Strong rarefaction waves; p-System with artificial viscosity; Global stability; Continuity argument; Positively invariant regions

Contents

1. Introduction and the statement of our main results

In this paper, we are concerned with the large time behavior of global smooth solutions to the Cauchy problem for the p-system with artificial viscosity

$$\begin{cases} v_t-u_x=\varepsilon_1 v_{xx},\\ u_t+p(v)_x=\varepsilon_2 u_{xx},\\ (v,u)|_{t=0}=(v_0,u_0)(x)\to(v_\pm,u_\pm) \quad \text{as } x\to\pm\infty. \end{cases} \tag{1.1}$$

Here ε_1 and ε_2 are positive constants which represent the viscosity coefficients and $p(v)$ is a smooth function defined on $v>0$ satisfying

$$p'(v)<0, \qquad p''(v)>0 \quad \text{for } v>0. \tag{1.2}$$

It is well known that the above problem is closely related to the Riemann problem of the p-system

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=0,\\ (v,u)|_{t=0}=(v_0^{\mathrm R},u_0^{\mathrm R})(x)=\begin{cases}(v_-,u_-), & x<0,\\ (v_+,u_+), & x>0.\end{cases} \end{cases} \tag{1.3}$$

And in this paper, we will concentrated on the case when the Riemann problem (1.3) admits a unique global weak (rarefaction wave) solution $(V^{\mathrm{R}}, U^{\mathrm{R}})(x/t)$ containing a rarefaction wave of the first family, denoted by $(V_1^{\mathrm{R}}, U_1^{\mathrm{R}})(x/t)$, and another of the second family denoted by $(V_2^{\mathrm{R}}, U_2^{\mathrm{R}})(x/t)$. That is, there exists a unique constant state $(v_m, u_m) \in \mathbf{R}^2$ $(v_m > 0)$ such that $(v_m, u_m) \in \mathbf{R}_1(v_-, u_-)$ and $(v_+, u_+) \in \mathbf{R}_2(v_m, u_m)$. Here

$$\begin{cases} \mathbf{R}_1(v_-, u_-) = \left\{(v, u) \mid u = u_- + \int_{v_-}^{v} \sqrt{-p'(z)}\, dz,\ u \geqslant u_-\right\}, \\ \mathbf{R}_2(v_m, u_m) = \left\{(v, u) \mid u = u_m - \int_{v_m}^{v} \sqrt{-p'(z)}\, dz,\ u \geqslant u_m\right\}. \end{cases} \tag{1.4}$$

In other words, the weak solution $(V^{\mathrm{R}}, U^{\mathrm{R}})(x/t)$ to the Riemann problem (1.3) is given by

$$\left(V^{\mathrm{R}}, U^{\mathrm{R}}\right)\left(\frac{x}{t}\right) = \left(V_1^{\mathrm{R}}\left(\frac{x}{t}\right) + V_2^{\mathrm{R}}\left(\frac{x}{t}\right) - v_m, U_1^{\mathrm{R}}\left(\frac{x}{t}\right) + U_2^{\mathrm{R}}\left(\frac{x}{t}\right) - u_m\right). \tag{1.5}$$

To study the above problem, as in [16,17], we first construct a smooth approximation to the above Riemann solution (1.5). Let $w_i(t, x)$ $(i = 1, 2)$ be the unique global smooth solution to the following Cauchy problem

$$\begin{cases} w_{it} + w_i w_{ix} = 0, \\ w_i(t, x)|_{t=0} = w_{i0}(x) = \frac{w_{i+} + w_{i-}}{2} + \frac{w_{i+} - w_{i-}}{2} K_q \int_0^x (1 + y^2)^{-q}\, dy, \end{cases} \tag{1.6}$$

where $q > \frac{3}{2}$, $K_q = (\int_0^x (1 + y^2)^{-q}\, dy)^{-1}$, and

$$\begin{cases} w_{1-} = \lambda_1(v_-) = -\sqrt{-p'(v_-)}, \\ w_{1+} = \lambda_1(v_m) = -\sqrt{-p'(v_m)}, \\ w_{2-} = \lambda_2(v_m) = \sqrt{-p'(v_m)}, \\ w_{2+} = \lambda_2(v_+) = \sqrt{-p'(v_+)}, \end{cases}$$

then $(V(t, x), U(t, x))$, the smooth approximation of the rarefaction waves profile, is defined by

$$(V, U)(t, x) = \left(\overline{V}_1(t + t_0, x) + \overline{V}_2(t + t_0, x) - v_m,\ \overline{U}_1(t + t_0, x) + \overline{U}_2(t + t_0, x) - u_m\right), \tag{1.7}$$

where $t_0 > 0$ is a sufficiently large but fixed positive constant which will be determined later and $(\overline{V}_i, \overline{U}_i)(t, x)$ $(i = 1, 2)$ are given implicitly by the following equations

$$\begin{cases} \lambda_i(\overline{V}_i(t, x)) = w_i(t, x),\ \lambda_i(v) = (-1)^i \sqrt{-p'(v)}, \quad i = 1, 2, \\ \overline{U}_1(t, x) = u_- + \int_{v_-}^{\overline{V}_1(t,x)} \sqrt{-p'(s)}\, ds, \\ \overline{U}_2(t, x) = u_m - \int_{v_m}^{\overline{V}_2(t,x)} \sqrt{-p'(s)}\, ds. \end{cases} \tag{1.8}$$

It is easy to check that $(V(t, x), U(t, x))$ solves

$$\begin{cases} V_t - U_x = 0, \\ U_t + p(V)_x = g(V)_x, \end{cases} \tag{1.9}$$

where

$$g(V)=p(V)-p(\overline{V}_1)-p(\overline{V}_2)+p(v_m). \tag{1.10}$$

The main purpose of our present paper is to compare the large time behavior of the global smooth solutions $(v(t,x),u(t,x))$ of the Cauchy problem (1.1) with the corresponding rarefaction wave solution $(V^{\mathrm{R}},U^{\mathrm{R}})(x/t)$ of the Riemann problem (1.3). That is, we want to study the nonlinear stability of the rarefaction wave solution $(V^{\mathrm{R}},U^{\mathrm{R}})(x/t)$ for the p-system (1.1) with artificial viscosity. Recall that the study on the nonlinear stability of rarefaction waves for dissipative hyperbolic conservation laws has a long history starting with the paper of A.M. Il'in and O.A. Oleinik [8] which studied the nonlinear stability of rarefaction waves for scalar viscous conservation laws by using the maximal principle. Since then, a lot of good results have been obtained for hyperbolic conservation laws with various dissipation terms by employing various methods (a complete list of the literature in this direction is beyond the scope of this paper, the interested reader is referred to [1–36] and the references cited therein). Roughly speaking, if the initial data is a small (in certain Sobolev norms) perturbation of the rarefaction wave profile constructed above, the results on the nonlinear stability of weak rarefaction waves are quite complete. As to the global stability of strong rarefaction waves for dissipative hyperbolic conservation laws, to the best of our knowledge, only the following results are available now: for scalar (even multidimensional) viscous conservation laws, the results are quite perfect, cf. [9,20,30] and the references cited therein. For systems with dissipative terms, some results are obtained only for the compressible Navier–Stokes equations [12,17,19,21] and the Jin–Xin relaxation model for the p-system [22].

Now let us back to our problem, that is for systems of hyperbolic conservation laws with artificial viscosity like (1.1), if the H^1-norm of the initial perturbation $(v_0(x)-V(0,x),u_0(x)-U(0,x))$ is small, the nonlinear stability of strong rarefaction waves is well-understood (cf. [24, 28,29,32]), but for the case when the $\|(v_0(x)-V(0,x),u_0(x)-U(0,x))\|_{H^1}$ is large, to our knowledge, fewer results have been obtained and the main purpose of our present paper is devoted to obtaining some results on the nonlinear stability of strong rarefaction wave solution $(V^{\mathrm{R}},U^{\mathrm{R}})(x/t)$ for the Cauchy problem (1.1) for the case when the H^1-norm of the initial perturbation, i.e., $\|(v_0(x)-V(0,x),u_0(x)-U(0,x))\|_{H^1}$ is large.

Now we give the main results obtained in this paper. Our analysis is based on the continuity argument and two types of results are obtained in this paper depending on whether the two positive constants ε_1 and ε_2 equal or not. Firstly, for the case $\varepsilon_1\neq\varepsilon_2$, to use the continuity argument to extend the local solution globally, we need to impose the following technical assumption on the initial perturbation $(v_0(x)-V(0,x),u_0(x)-U(0,x))$: There exist three constants $\beta_2\geqslant 0$, $0<\beta_1<q_1=\min\{\frac{1}{4q},\frac{2q}{3}-1\}$ such that

$$\begin{cases}\|(v_0(x)-V(0,x),\ u_0(x)-U(0,x))\|^2\leqslant D_1t_0^{-q_1},\\ \|(v_{0x}(x)-V_x(0,x))\|^2\leqslant D_2(1+t_0^{\beta_1}),\\ \|(u_{0x}(x)-U_x(0,x))\|^2\leqslant D_3(1+t_0^{\beta_2}).\end{cases} \tag{1.11}$$

Here D_i $(i=1,2,3)$ are some positive constants independent of t_0. And our first result can be stated as follows:

Theorem 1.1. *Suppose that the smooth nonlinear function $p(v)$ satisfies* (1.2) *and that the initial disturbance $(v_0(x) - V(0,x),\ u_0(x) - U(0,x)) \in H^1(\mathbf{R})$. If we assume further that there exist constants $\delta > 0$, $q_1 = \min\{\frac{1}{4q}, \frac{2q}{3} - 1\} > 0$, $\beta_1 > 0$, $\beta_2 \geqslant 0$ such that $\beta_1 < q_1$, $v_0(x) \geqslant \delta > 0$ and* (1.11) *is satisfied, then the Cauchy problem* (1.1) *admits a unique global smooth solution $(v(t,x), u(t,x))$ which satisfies*

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left| \left(v(t,x) - V^{\mathrm{R}}\left(\frac{x}{t}\right), u(t,x) - U^{\mathrm{R}}\left(\frac{x}{t}\right) \right) \right| = 0. \tag{1.12}$$

Remark 1.1. Since in the proof of Theorem 1.1, $t_0 > 0$ is assumed to be a sufficiently large constant, thus although the H^1-norm of the initial perturbation $(v_0(x) - V(0,x), u_0(x) - U(0,x))$ can be large in Theorem 1.1, the assumption $(1.11)_1$ does imply that the L^2-norm of the initial perturbation is small. In some sense it is a generalization of the result obtained in [5] for the case $(v_-, u_-) = (v_+, u_+)$ to the case $(v_-, u_-) \neq (v_+, u_+)$. Moreover it is worth to pointing out that to use the method developed by Y. Kanel' in [11] to deduce the uniform lower bound for $v(t,x)$, we need to ask that the positive constants β_1 and q_1 satisfy $\beta_1 < q_1$.

Remark 1.2. Since $q > \frac{3}{2}$, we have $q_1 < \frac{1}{6}$ and it is easy to see that if we take q sufficiently large, $q_1 = \frac{1}{4q}$.

Secondly, when $\varepsilon_1 = \varepsilon_2$, (1.1) admits a positively invariant region (cf. [1,23]), from which we can get the uniform lower bound for $v(t,x)$. With this estimate in hand, if we ask the initial disturbance to satisfy

$$\left\| \left(v_0(x) - V(0,x),\ u_0(x) - U(0,x) \right) \right\|_{H^1} \leqslant C_0 + \frac{\sqrt{\kappa^{-1}(t_0^{\alpha})}}{8C_1} \tag{1.13}$$

for each $0 < \alpha < \frac{1}{8q}$, $C_1 = \max\{v_-, v_m, v_+, C_4, C_5\}$, $C_4 > 1$ and $C_5 > 1$ are positive constants defined by (4.2) and (4.11), respectively, and $\kappa^{-1}(x)$ denotes the inverse function of $\kappa(x)$, a strictly increasing, continuous function defined in Section 4, then we have:

Theorem 1.2. *Suppose that $\varepsilon_1 = \varepsilon_2$, $p(v)$ satisfies* (1.2) *and that there exists a positive constant $\delta > 0$ such that $v_0(x) \geqslant \delta$. Then the Cauchy problem* (1.1) *admits a unique global smooth solution $(v(t,x), u(t,x))$ satisfying $v(t,x) \geqslant \delta$ and* (1.12) *provided that* (1.13) *is satisfied.*

Remark 1.3. It is easy to see that $\kappa^{-1}(t_0^{\alpha}) \to +\infty$ as $t_0 \to +\infty$. Note also the assumption we imposed on the range of the exponent, i.e., $\alpha \in (0, \frac{1}{8q})$ in (1.13) is not optimal.

Although we do not ask the initial perturbation to be small in Theorem 1.2, the assumption (1.13) does indicate that if the H^1-norm of the initial perturbation depends on t_0, some restrictions on its growth rate as $t_0 \to \infty$ must be imposed. However, our last result in this paper is to show that if the smooth function $p(v)$ satisfies some growth conditions as $v \to \infty$, which contains the typical case of $p(v)$, i.e., $p(v) = v^{-\gamma}$ $(\gamma \geqslant 1)$ as special example, we can show the global stability of the rarefaction wave for any H^1-perturbation. In fact, if we assume further that $p(v)$ satisfies that for $v \geqslant \delta > 0$

$$\begin{cases} \frac{|p(v)-p(V)|}{\sqrt{\Phi(V,v-V)}} \leqslant O(1), \\ \left|\frac{p''(v)}{p'(v)}\right| \leqslant O(1), \\ \frac{|v-V||p'(v)-p'(V)|}{\Phi(V,v-V)} \leqslant O(1), \\ \frac{|v-V||p''(v)-p''(V)|}{\Phi(V,v-V)} \leqslant O(1), \\ \frac{|v-V||p''(v)|}{\sqrt{-p'(v)}\sqrt{\Phi(V,v-V)}} \leqslant O(1), \end{cases} \tag{1.14}$$

where $\Phi(V, Z) = p(V)Z - \int_V^{V+Z} p(s)\,ds$, then we have the following theorem.

Theorem 1.3. *Suppose that $\varepsilon_1 = \varepsilon_2$, $p(v)$ satisfies* (1.2) *and that there exists a positive constant $\delta > 0$ such that $v_0(x) \geqslant \delta$. Then the same result stated in Theorem* 1.1 *holds for any $(v_0(x) - V(0,x), u_0(x) - U(0,x)) \in H^1(\mathbf{R})$ provided that* (1.14) *is satisfied.*

Before concluding this section, we outline the main ingredients used in proving our main results. Our analysis is based on the continuity argument together with an elementary energy method. Another ingredient of our method is to introduce the quantity $t_0 > 0$ in the construction of smooth approximation to the Riemann solution $(V^{\mathrm{R}}, U^{\mathrm{R}})(x/t)$ to control the possible growth caused by the nonlinearity of (1.1) and/or by the interactions of waves from different families.

This paper is arranged as follows: we will give some basic estimates in Section 2. The proofs of Theorems 1.1, 1.2 and 1.3 are then given in Sections 3, 4 and 5, respectively.

Throughout the rest of this paper, C, D or $O(1)$ will be used to denote a generic positive constant independent of t, t_0 and x and $C_i(\cdot,\cdot)$ or $D_i(\cdot,\cdot)$ $(i \in \mathbf{Z}_+)$ stands for some generic constants depending only on the quantities listed in the parenthesis. For two functions $f(x) \sim g(x)$ as $x \to a$ means that there exists a positive constant $C > 0$ such that $C^{-1}f(x) \leqslant g(x) \leqslant Cf(x)$ in the neighborhood of a. $H^l(\mathbf{R})$ $(l \geqslant 0)$ denotes the usual Sobolev space with norm $\|\cdot\|_l$ and $\|\cdot\|_0 = \|\cdot\|$ will be used to denote the usual L^2-norm. For a vector $a = (a_1, a_2, \ldots, a_n)$, $|a| = (\sum_{i=1}^n a_i^2)^{1/2}$ and for $1 \leqslant p \leqslant +\infty$,

$$f(x) \in L^p(\mathbf{R}, \mathbf{R}^n), \quad |f|_p = \left(\int_{\mathbf{R}} |f(x)|^p\,dx\right)^{1/p}.$$

It is easy to see that $|f|_2 = \|\cdot\|$. Finally, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{L^\infty_{t,x}}$ are used to denote $\|\cdot\|_{L^\infty(\mathbf{R})}$ and $\|\cdot\|_{L^\infty([0,t]\times\mathbf{R})}$, respectively.

2. Preliminaries

In this section, we give some basic estimates which will be used in proving our main results.

Firstly we list some basic properties of the global smooth functions $(V(t,x), U(t,x))$ constructed in (1.7) and (1.8) whose proof can be found in [17].

Lemma 2.1. *The smooth functions $(V(t,x), U(t,x))$ constructed in* (1.7) *and* (1.8) *satisfies*

(i) $V_t(t,x) = U_x(t,x) > 0$ *for all* $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$;

(ii) *For any* $p \in [1, +\infty]$*, there exists a positive constant* $C(p,q)$ *such that*

$$\begin{cases} |(V_x, U_x)(t)|_p \leqslant C(p,q)(t+t_0)^{-1+\frac{1}{p}}, \\ |(V_{xx}, U_{xx})(t)|_p \leqslant C(p,q)\min\big\{(t+t_0)^{-1-\frac{p-1}{2pq}}, (t+t_0)^{-2+\frac{1}{p}}\big\}, \\ |g(V)_x(t)|_p \leqslant C(p,q)(t+t_0)^{-\frac{2q}{3}}. \end{cases}$$

Especially

$$\begin{cases} \int_0^\infty |(V_{xx}, U_{xx})|_p\,dt \leqslant C(p,q)\min\big\{t_0^{-\frac{p-1}{2pq}}, t_0^{-1+\frac{1}{p}}\big\}, & \textit{for } p > 1, \\ \int_0^\infty |g(V)_x(t)|_p\,dt \leqslant C(p,q)t_0^{1-\frac{2q}{3}}, & \textit{for } p \geqslant 1,\ q > \frac{3}{2}; \end{cases}$$

(iii) $\lim_{t\to+\infty}\sup_{x\in\mathbf{R}}|(V,U)(t,x) - (V^{\mathrm{R}}, U^{\mathrm{R}})(\frac{x}{t})| = 0$;
(iv) $|(V_x, U_x)(t,x)| \leqslant O(1)|(V_t, U_t)(t,x)|$.

Now we turn to some estimates related to $p(v)$ and on the solutions $(v(t,x), u(t,x))$ to the Cauchy problem (1.1). Let

$$\begin{cases} \phi(t,x) = v(t,x) - V(t,x), \\ \psi(t,x) = u(t,x) - U(t,x), \end{cases} \tag{2.1}$$

then it is easy to verify that $(\phi(t,x), \psi(t,x))$ solves

$$\begin{cases} \phi_t - \psi_x = \varepsilon_1 V_{xx} + \varepsilon_1 \phi_{xx}, \\ \psi_t + (p(V+\phi) - p(V))_x = \varepsilon_2 U_{xx} + \varepsilon_2 \psi_{xx} - g(V)_x, \\ (\phi(0,x), \psi(0,x)) = (\phi_0(x), \psi_0(x)) = (v_0(x) - V(0,x), u_0(x) - U(0,x)). \end{cases} \tag{2.2}$$

Next we present a useful lemma whose proof can be found in [21].

Lemma 2.2. *If* $p(v)$ *satisfies* (1.2)*, then there exists a positive constant* $C_2 > 0$ *such that*

$$\Phi(V, v-V) \geqslant C_2 \frac{(v-V)^2}{v+V}, \quad v \in (0,\infty). \tag{2.3}$$

We now turn to consider the reformulated problem (2.2). We will seek its solutions in the following function space

$$\begin{aligned} &X_{\delta,M}(0,T) \\ &\quad = \Big\{(\phi,\psi)(t,x)\colon\ (\phi,\psi)(t,x) \in C\big([0,T]; H^1(\mathbf{R})\big), (\phi_x, \psi_x)(t,x) \in L^2\big([0,T]; H^1(\mathbf{R})\big), \\ &\qquad \sup_{t\in[0,T]} \big\|(\phi,\psi)(t)\big\|_1 \leqslant M,\ \inf_{[0,T]\times\mathbf{R}}\big(V(t,x) + \phi(t,x)\big) \geqslant \delta\Big\}. \end{aligned} \tag{2.4}$$

To this end, the iteration scheme

$$\begin{cases} (\phi^0, \psi^0)(t,x) = (\phi_0, \psi_0)(x), \\ \psi_t^{k+1} - \varepsilon_2 \psi_{xx}^{k+1} = F(\phi^k) = \varepsilon_2 U_{xx} - g(V)_x + (p(V) - p(V+\phi^k))_x, \\ \phi_t^{k+1} - \varepsilon_1 \phi_{xx}^{k+1} = G(\psi^{k+1}) = \varepsilon_1 V_{xx} + \psi_x^{k+1} \end{cases} \tag{2.5}$$

together with a standard contracting-mapping argument yield the following local existence theorem.

Lemma 2.3 *(Local existence). If* $\|(\phi_0, \psi_0)(x)\|_1 \leqslant M$ *and* $\inf_{x\in\mathbf{R}}(V(0,x) + \phi_0(x)) \geqslant \delta$, *then there exists* $t_1 = t_1(\delta, M) > 0$ *such that the Cauchy problem* (2.2) *admits a unique smooth solution* $(\phi, \psi) \in X_{\frac{1}{2}\delta, 2M}(0, t_1)$ *on the strip* $\Pi(0, t_1) = \{(t,x)\colon\ 0 \leqslant t \leqslant t_1,\ x \in \mathbf{R}\}$. *Moreover, for all* $0 \leqslant t \leqslant t_1$, $(\phi(t,x), \psi(t,x))$ *satisfies*

$$\begin{cases} \|(\phi(t,x), \psi(t,x))\|^2 \leqslant 2\|(\phi_0(x), \psi_0(x))\|^2, \\ \|\phi_x(t,x)\|^2 \leqslant 2\|\phi_{0x}(x)\|^2, \\ \|\psi_x(t,x)\|^2 \leqslant 2\|\psi_{0x}(x)\|^2. \end{cases} \tag{2.6}$$

Finally, when $\varepsilon_1 = \varepsilon_2$, by exploiting the theory of positively invariant regions for reaction diffusion equations developed by Chueh, Conley and Smoller in [1], we know that the region

$$\Sigma = \left\{ (v,u) \mid r = u - \int^v \sqrt{-p'(\xi)}\, d\xi \leqslant r_0, s = u + \int^v \sqrt{-p'(\xi)}\, d\xi \geqslant s_0 \right\} \tag{2.7}$$

is a positively invariant region for (1.1). Thus we can get from the assumption (1.2) we imposed on $p(v)$ that

Lemma 2.4 *(Positively invariant region). Assume that* $\varepsilon_1 = \varepsilon_2$, $p(v)$ *satisfies* (1.2) *and* $(v(t,x), u(t,x))$ *is a smooth solution to the Cauchy problem* (1.1) *defined on the strip* $\Pi(0,T) = \{(t,x)\colon\ 0 \leqslant t \leqslant T,\ x \in \mathbf{R}\}$. *If we assume further that there exists a constant* $\delta > 0$ *such that* $v_0(x) \geqslant \delta$, *then we can deduce that* $v(t,x) \geqslant \delta$ *for all* $0 \leqslant t \leqslant T$, $x \in \mathbf{R}$.

3. The proof of Theorem 1.1

This section is devoted to prove Theorem 1.1. To do so, we first give the following energy estimates.

Lemma 3.1 *(A priori estimate). Let* $(\phi(t,x), \psi(t,x)) \in X_{\bar{\delta}, \overline{M}}(0,T)$ *be a solution to the Cauchy problem* (2.2) *defined on the strip* $\Pi(0,T) = \{(t,x)\colon\ 0 \leqslant t \leqslant T,\ x \in \mathbf{R}\}$. *Then if we can choose* t_0 *sufficiently large such that*

$$\begin{cases} C_1(\bar{\delta}, \overline{M}) t_0^{-\frac{1}{8q}} \leqslant 1, \\ O(1) C_2(\bar{\delta}, \overline{M}) t_0^{-1} < \frac{1}{2}, \end{cases} \tag{3.1}$$

we have for $0 \leqslant t \leqslant T$ *that*

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\frac{1}{2}\psi^2 + \Phi(V, v - V)\right) dx + \varepsilon_2 \int_0^t \int_{\mathbf{R}} \psi_x^2 \, dx \, d\tau \\
&\quad + \frac{\varepsilon_1}{2} \int_0^t \int_{\mathbf{R}} \big(- p'(V+\phi)\phi_x^2 \big) \, dx \, d\tau + \int_0^t \int_{\mathbf{R}} \big(p(V+\phi) - p(V) - p'(V)\phi \big) V_t \, dx \, d\tau \\
&\leqslant O(1)\big(t_0^{-q_1} + \big\|(\phi_0, \psi_0)(x)\big\|^2\big),
\end{aligned} \tag{3.2}$$

$$\int_{\mathbf{R}} \phi_x^2 \, dx + \varepsilon_1 \int_0^t \int_{\mathbf{R}} \phi_{xx}^2 \, dx \, d\tau \leqslant O(1)\big(t_0^{-q_1} + \big\|(\phi_0, \psi_0)(x)\big\|^2\big) + O(1)\big\|\phi_{0x}(x)\big\|^2 \tag{3.3}$$

and

$$\int_{\mathbf{R}} \psi_x^2 \, dx + \varepsilon_2 \int_0^t \int_{\mathbf{R}} \psi_{xx}^2 \, dx \, d\tau \leqslant O(1)\big|p'(\bar{\delta})\big|\big(t_0^{-q_1} + \big\|(\phi_0, \psi_0)(x)\big\|^2\big) + O(1)\big\|\psi_{0x}(x)\big\|^2. \tag{3.4}$$

Here $q_1 = \min\{\frac{1}{4q}, \frac{2q}{3} - 1\} > 0$, *and*

$$\begin{cases} C_1(\bar{\delta}, \overline{M}) = \big(\sup_{\bar{\delta} \leqslant v \leqslant \overline{M}} \frac{|p(V) - p(V+\phi)|}{\sqrt{\Phi(V, v-V)}}\big)^2, \\ C_2(\bar{\delta}, \overline{M}) = \big(\sup_{\bar{\delta} \leqslant v \leqslant \overline{M}} \frac{|p'(V) - p'(V+\phi)|}{\sqrt{-p'(V+\phi)}\sqrt{p(V+\phi) - p(V) - p'(V)\phi}}\big)^2. \end{cases} \tag{3.5}$$

Proof. Multiplying $(2.2)_1$ by $p(V) - p(V+\phi)$ and $(2.2)_2$ by ψ, adding the resulting two identities, and integrating the final result w.r.t. t and x over $[0, t] \times \mathbf{R}$, we have

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\frac{1}{2}\psi^2 + \Phi(V, v - V)\right) dx \bigg|_0^t + \int_0^t \int_{\mathbf{R}} \big(p(V+\phi) - p(V) - p'(V)\phi \big) V_t \, dx \, d\tau \\
&\quad + \varepsilon_2 \int_0^t \int_{\mathbf{R}} \psi_x^2 \, dx \, d\tau + \varepsilon_1 \int_0^t \int_{\mathbf{R}} \big(- p'(V+\phi)\big)\phi_x^2 \, dx \, d\tau \\
&= \int_0^t \int_{\mathbf{R}} \varepsilon_1 \big(p(V) - p(V+\phi) \big) V_{xx} \, dx \, d\tau + \varepsilon_1 \int_0^t \int_{\mathbf{R}} \big(p'(V+\phi) - p'(V) \big) V_x \phi_x \, dx \, d\tau \\
&\quad + \varepsilon_2 \int_0^t \int_{\mathbf{R}} U_{xx} \psi \, dx \, d\tau - \int_0^t \int_{\mathbf{R}} g(V)_x \psi \, dx \, d\tau = \sum_{j=1}^4 I_j,
\end{aligned} \tag{3.6}$$

where I_j, $j = 1, 2, 3, 4$, are the corresponding terms in the above equation.

Since $0 < \bar{\delta} \leqslant v \leqslant \overline{M}$, from Lemma 2.1 and the assumptions we imposed on $p(v)$, I_j $(j = 1, 2, 3, 4)$ can be estimated as in the following:

$$\begin{aligned}
|I_1| &= \int_0^t \int_{\mathbf{R}} \frac{|p(V) - p(V+\phi)|}{\sqrt{\Phi(V, v-V)}} \sqrt{\Phi(V, v-V)}\, \varepsilon_1 |V_{xx}|\, dx\, d\tau \\
&\leqslant \sup_{\bar{\delta} \leqslant v \leqslant \overline{M}} \left(\frac{|p(V) - p(V+\phi)|}{\sqrt{\Phi(V, v-V)}} \right) \varepsilon_1 \int_0^t \int_{\mathbf{R}} \sqrt{\Phi(V, v-V)}\, |V_{xx}|\, dx\, d\tau \\
&\leqslant \sup_{\bar{\delta} \leqslant v \leqslant \overline{M}} \left(\frac{|p(V) - p(V+\phi)|}{\sqrt{\Phi(V, v-V)}} \right) \varepsilon_1 \int_0^t \big\| \sqrt{\Phi} \big\| \|V_{xx}\|\, d\tau \\
&\leqslant \varepsilon_1^2 \left(\sup_{\bar{\delta} \leqslant v \leqslant \overline{M}} \frac{|p(V) - p(V+\phi)|}{\sqrt{\Phi(V, v-V)}} \right)^2 \int_0^t \|V_{xx}\| \big\| \sqrt{\Phi} \big\|^2 d\tau + O(1) \int_0^t \|V_{xx}\|\, d\tau \\
&\leqslant \varepsilon_1^2 C_1(\delta, M) t_0^{-\frac{1}{8q}} \int_0^t (1+\tau)^{-1-\frac{1}{8q}} \big\| \sqrt{\Phi} \big\|^2 d\tau + O(1) t_0^{-\frac{1}{4q}},
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
|I_2| &\leqslant \frac{\varepsilon_1}{2} \int_0^t \int_{\mathbf{R}} \big(-p'(V+\phi)\big) \phi_x^2\, dx\, d\tau \\
&\quad + O(1) C_2(\bar{\delta}, \overline{M}) \|V_x\|_{L^\infty_{t,x}} \int_0^t \int_{\mathbf{R}} \big(p(V+\phi) - p(V) - p'(V)\phi\big) V_t\, dx\, d\tau \\
&\leqslant \frac{\varepsilon_1}{2} \int_0^t \int_{\mathbf{R}} \big(-p'(V+\phi)\big) \phi_x^2\, dx\, d\tau \\
&\quad + O(1) C_2(\bar{\delta}, \overline{M}) t_0^{-1} \int_0^t \int_{\mathbf{R}} \big(p(V+\phi) - p(V) - p'(V)\phi\big) V_t\, dx\, d\tau
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
|I_3 + I_4| &\leqslant \int_0^t \|\psi\| \big(\varepsilon_2 \|U_{xx}\| + \big\| g(v)_x \big\|\big)\, d\tau \\
&\leqslant \int_0^t \|\psi\|^2 \big(\varepsilon_2 \|U_{xx}\| + \big\| g(V)_x \big\|\big)\, d\tau + \int_0^t \big(\varepsilon_2 \|U_{xx}\| + \big\| g(V)_x \big\|\big)\, d\tau
\end{aligned}$$

$$\leqslant \int_0^t \|\psi\|^2\big(\varepsilon_2\|U_{xx}\| + \|g(V)_x\|\big)\,d\tau + O(1)t_0^{-q_1}$$

$$\leqslant \int_0^t (1+\tau)^{-1-q_1}\|\psi\|^2\,d\tau + O(1)t_0^{-q_1}. \tag{3.9}$$

Inserting (3.7)–(3.9) into (3.6), we arrive at

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\frac{1}{2}\psi^2 + \Phi(V, v-V)\right)dx\bigg|_0^t \\
&\quad + \big(1 - O(1)C_2(\bar{\delta}, \overline{M})t_0^{-1}\big)\int_0^t\int_{\mathbf{R}} \big(p(V+\phi) - p(V) - p'(V)\phi\big)V_t\,dx\,d\tau \\
&\quad + \varepsilon_2\int_0^t\int_{\mathbf{R}} \psi_x^2\,dx\,d\tau + \frac{\varepsilon_1}{2}\int_0^t\int_{\mathbf{R}} \big(-p'(V+\phi)\big)\phi_x^2\,dx\,d\tau \\
&\leqslant O(1)t_0^{-q_1} + O(1)C_1(\bar{\delta}, \overline{M})t_0^{-1-q_2}\int_0^t (1+\tau)^{-1-\frac{1}{8q}}\Big\|\Big(\sqrt{\Phi(V, v-V)}, \psi\Big)\Big\|^2\,d\tau.
\end{aligned} \tag{3.10}$$

Here $q_2 = \min\{\frac{1}{8q}, \frac{2q}{3} - 1\}$.

Having obtained (3.10), it is easy to see that if we can choose t_0 sufficiently large such that (3.1) is satisfied, then by Gronwall's inequality we can get (3.2) immediately.

Next, we prove (3.3). For this purpose, we multiply $(2.2)_1$ by $-\phi_{xx}$ and integrate the result w.r.t. t and x over $[0, t] \times \mathbf{R}$ to get that

$$\int_{\mathbf{R}} \frac{\phi_x^2}{2}\,dx\bigg|_0^t + \varepsilon_1\int_0^t\int_{\mathbf{R}} \phi_{xx}^2\,dx\,d\tau = -\varepsilon_1\int_0^t\int_{\mathbf{R}} V_{xx}\phi_{xx}\,dx\,d\tau - \int_0^t\int_{\mathbf{R}} \psi_x\phi_{xx}\,dx\,d\tau = \sum_{j=5}^{6} I_j. \tag{3.11}$$

Due to

$$|I_5| \leqslant \frac{\varepsilon_1}{2}\int_0^t\int_{\mathbf{R}} \phi_{xx}^2\,dx\,d\tau + \frac{\varepsilon_1}{2}\int_0^t\int_{\mathbf{R}} V_{xx}^2\,dx\,d\tau \leqslant \frac{\varepsilon_1}{2}\int_0^t\int_{\mathbf{R}} \phi_{xx}^2\,dx\,d\tau + O(1)t_0^{-\frac{1}{4q}} \tag{3.12}$$

and

$$|I_6| \leqslant \frac{\varepsilon_1}{4}\int_0^t\int_{\mathbf{R}} \phi_{xx}^2\,dx\,d\tau + O(1)\int_0^t\int_{\mathbf{R}} \psi_x^2\,dx\,d\tau, \tag{3.13}$$

(3.3) follows immediately from (3.2), (3.11)–(3.13).

At last for (3.4), we have by multiplying $(2.2)_2$ by $-\psi_{xx}$ and integrating the resulting identity w.r.t. t and x over $[0,t]\times\mathbf{R}$ to get that

$$\begin{aligned}
&\int_{\mathbf{R}} \frac{\psi_x^2}{2}\,dx\bigg|_0^t + \varepsilon_2\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau \\
&\quad = -\varepsilon_2\int_0^t\int_{\mathbf{R}} U_{xx}\psi_{xx}\,dx\,d\tau + \int_0^t\int_{\mathbf{R}} g(V)_x\psi_{xx}\,dx\,d\tau + \int_0^t\int_{\mathbf{R}} p'(V+\phi)\phi_x\psi_{xx}\,dx\,d\tau \\
&\qquad + \int_0^t\int_{\mathbf{R}} \big(p'(V+\phi)-p'(V)\big)V_x\psi_{xx}\,dx\,d\tau = \sum_{j=7}^{10} I_j.
\end{aligned} \tag{3.14}$$

Since

$$\begin{aligned}
|I_7+I_8| &\leqslant \frac{\varepsilon_2}{2}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + \frac{1}{2}\int_0^t\int_{\mathbf{R}} \big(V_{xx}^2 + g(V)_x^2\big)\,dx\,d\tau \\
&\leqslant \frac{\varepsilon_2}{4}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + O(1)t_0^{-\frac{1}{4q}},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
|I_9| &\leqslant \frac{\varepsilon_2}{4}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + O(1)\big|p'(\bar\delta)\big|\int_0^t\int_{\mathbf{R}} \big(-p'(V+\phi)\big)\phi_x^2\,dx\,d\tau \\
&\leqslant \frac{\varepsilon_2}{4}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + O(1)\big|p'(\bar\delta)\big|\big(t_0^{-q_1} + \big\|(\phi_0,\psi_0)(x)\big\|^2\big)
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
|I_{10}| &\leqslant \frac{\varepsilon_2}{8}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + O(1)C_2(\bar\delta,\overline{M})t_0^{-1}\int_0^t\int_{\mathbf{R}} \big(p(V+\phi)-p(v)-p'(V)\phi\big)V_t\,dx\,d\tau \\
&\leqslant \frac{\varepsilon_2}{8}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau + O(1)\big(t_0^{-q_1} + \big\|(\phi_0,\psi_0)(x)\big\|^2\big).
\end{aligned} \tag{3.17}$$

Inserting (3.15)–(3.17) into (3.14), we arrive at

$$\int_{\mathbf{R}} \frac{\psi_x^2}{2}\,dx + \frac{\varepsilon_2}{8}\int_0^t\int_{\mathbf{R}} \psi_{xx}^2\,dx\,d\tau \leqslant O(1)\big|p'(\bar\delta)\big|\big(t_0^{-q_1} + \big\|(\phi_0,\psi_0)(x)\big\|^2\big) + \big\|\psi_{0x}(x)\big\|^2. \tag{3.18}$$

This is (3.4) and the proof of Lemma 3.1 is completed. □

The next lemma is to get an estimate on $\int_{\mathbf{R}} \frac{\phi_x^2}{v^2}\,dx$ which will play an important role in deducing the uniform lower bound for $v(t,x)$.

Lemma 3.2. *Under the assumption of Lemma* 3.1, *we have*

$$\begin{aligned}
&\left\|\frac{\phi_x}{v}\right\|^2+\int_0^t\left\|\frac{\phi_{xx}}{v}\right\|^2 d\tau\\
&\quad\leqslant O(1)\Big\{\|\phi_{0x}(x)\|^2+C_3(\bar\delta,\overline{M})\big(t_0^{-q_1}+\big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big)\Big\}\\
&\qquad\times\exp\Big\{O(1)C_3(\bar\delta,\overline{M})\big[t_0^{\gamma}\big(t_0^{-q_1}+\big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big)+t_0^{-\gamma}\|\phi_{0x}\|^2\big]\Big\}. \qquad (3.19)
\end{aligned}$$

Here $\gamma>0$ is some constant and $C_3(\bar\delta,\overline{M})$ is defined by

$$C_3(\bar\delta,\overline{M})=\max\left\{\bar\delta^{-2},\frac{\bar\delta^{-2}}{|p'(\overline{M})|},\frac{\bar\delta^{-8}}{|p'(\overline{M})|},\frac{\bar\delta^{-3}}{\sqrt{|p'(\overline{M})|}},\frac{\bar\delta^{-4}}{|p'(\overline{M})|},\frac{\bar\delta^{-3}}{|p'(\overline{M})|}\right\}.$$

Proof. Multiplying $(2.2)_1$ by $-\frac{\phi_{xx}}{v^2}$ and integrating the result w.r.t. t and x over $[0,t]\times\mathbf{R}$, we have:

$$\begin{aligned}
&\frac12\int_{\mathbf{R}}\frac{\phi_x^2}{v^2}\,dx\bigg|_0^t+\varepsilon_1\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau\\
&\quad=\int_0^t\int_{\mathbf{R}}\left(\frac{\phi_x^2\phi_t}{v^3}+\frac{2\phi_x\phi_t V_x-\phi_x^2V_t}{v^3}-\varepsilon_1\frac{\phi_{xx}V_{xx}}{v^2}-\frac{\phi_{xx}\psi_x}{v^2}\right)dx\,d\tau=\sum_{j=11}^{14}I_j. \qquad (3.20)
\end{aligned}$$

We now estimate I_j $(j=11,12,13,14)$ term by term in the following. Firstly, for I_{11} notice that $\phi_t=\psi_x+\varepsilon_1V_{xx}+\varepsilon_1\phi_{xx}$, we can get from (3.2)–(3.4) that

$$\begin{aligned}
I_{11}&=\int_0^t\int_{\mathbf{R}}\frac{\phi_x^2}{v^3}(\psi_x+\varepsilon_1V_{xx}+\varepsilon_1\phi_{xx})\,dx\,d\tau\\
&=\int_0^t\int_{\mathbf{R}}\left(\frac{\phi_x^2\psi_x}{v^3}+\varepsilon_1\frac{\phi_x^2V_{xx}}{v^3}+\varepsilon_1\frac{\phi_x^4}{v^4}+\varepsilon_1\frac{\phi_x^3V_x}{v^4}\right)dx\,d\tau\\
&\leqslant O(1)\int_0^t\int_{\mathbf{R}}\left(\frac{\phi_x^4}{v^4}+\frac{\psi_x^2}{v^2}+\frac{V_{xx}^2}{v^2}+\frac{|\phi_x|^3|V_x|}{v^4}\right)dx\,d\tau. \qquad (3.21)
\end{aligned}$$

Since

$$\int_0^t\int_{\mathbf{R}}\frac{\psi_x^2}{v^2}\,dx\,d\tau\leqslant\bar{\delta}^{-2}\int_0^t\int_{\mathbf{R}}\psi_x^2\,dx\,d\tau\leqslant O(1)\bar{\delta}^{-2}\big(t_0^{-q_1}+\big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big),\tag{3.22}$$

$$\int_0^t\int_{\mathbf{R}}\frac{V_{xx}^2}{v^2}\,dx\,d\tau\leqslant\bar{\delta}^{-2}\int_0^t\int_{\mathbf{R}}V_{xx}^2\,dx\,d\tau\leqslant O(1)\bar{\delta}^{-2}\big(t_0^{-q_1}+\big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big),\tag{3.23}$$

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\frac{\phi_x^4}{v^4}\,dx\,d\tau\\
&\quad\leqslant\int_0^t\left\|\frac{\phi_x}{v}\right\|_{L^\infty}^2\left\|\frac{\phi_x}{v}\right\|^2d\tau\leqslant\bar{\delta}^{-2}\int_0^t\big(t_0^{\gamma}\|\phi_x\|^2+t_0^{-\gamma}\|\phi_{xx}\|^2\big)\left\|\frac{\phi_x}{v}\right\|^2d\tau\\
&\quad\leqslant O(1)\bar{\delta}^{-2}\int_0^t\left(t_0^{\gamma}\sup_{\bar{\delta}\leqslant v\leqslant\overline{M}}\left(\frac{1}{|p'(v)|}\right)\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2+t_0^{-\gamma}\|\phi_{xx}\|^2\right)\left\|\frac{\phi_x}{v}\right\|^2d\tau\\
&\quad\leqslant O(1)C_3(\bar{\delta},\overline{M})\int_0^t\big(t_0^{\gamma}\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2+t_0^{-\gamma}\|\phi_{xx}\|^2\big)\left\|\frac{\phi_x}{v}\right\|^2d\tau
\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{|\phi_x|^3|V_x|}{v^4}\,dx\,d\tau&\leqslant\int_0^t\left\|\frac{\phi_x^2}{v^3}\right\|_{L^\infty}\left\|\frac{\phi_x}{v}\right\|\|V_x\|\,d\tau\leqslant O(1)\bar{\delta}^{-3}t_0^{-\frac12}\int_0^t\|\phi_x\|_{L^\infty}^2\left\|\frac{\phi_x}{v}\right\|d\tau\\
&\leqslant\frac{O(1)\bar{\delta}^{-4}}{\sqrt{|p'(\overline{M})|}}t_0^{-\frac12}\int_0^t\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|\left\|\frac{\phi_{xx}}{v}\right\|\left\|\frac{\phi_x}{v}\right\|d\tau\\
&\leqslant\frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau+O(1)C_3(\bar{\delta},\overline{M})\int_0^t\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2\left\|\frac{\phi_x}{v}\right\|^2d\tau,
\end{aligned}\tag{3.25}$$

we have from (3.21)–(3.25) that

$$\begin{aligned}
I_{11}&\leqslant O(1)C_3(\bar{\delta},\overline{M})\big(t_0^{-q_1}+\big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big)+\frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau\\
&\quad+O(1)C_3(\bar{\delta},\overline{M})\int_0^t\big(t_0^{\gamma}\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2+t_0^{-\gamma}\|\phi_{xx}\|^2\big)\left\|\frac{\phi_x}{v}\right\|^2d\tau.
\end{aligned}\tag{3.26}$$

Next for I_{13} and I_{14}, we have from Lemma 3.1 and Cauchy–Schwarz's inequality that

$$\begin{aligned}
I_{13} &\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{V_{xx}^2}{v^2}\,dx\,d\tau \\
&\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)C_3(\bar{\delta},\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big), \qquad (3.27)\\
I_{14} &\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\psi_x^2}{v^2}\,dx\,d\tau \\
&\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)C_3(\bar{\delta},\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big). \qquad (3.28)
\end{aligned}$$

Finally, for I_{12}, since $\phi_t = \psi_x + \varepsilon_1 V_{xx} + \varepsilon_1\phi_{xx}$, we have

$$\begin{aligned}
I_{12} &= \int_0^t\int_{\mathbf{R}}\frac{2\phi_x\phi_t V_x - \phi_x^2 V_t}{v^3}\,dx\,d\tau \\
&= \int_0^t\int_{\mathbf{R}}\left(\frac{2\phi_x\psi_x V_x}{v^3} + \frac{2\varepsilon_1\phi_x V_x V_{xx}}{v^3} + \frac{2\varepsilon_1\phi_x V_x\phi_{xx}}{v^3} - \frac{\phi_x^2 V_t}{v^3}\right)dx\,d\tau. \qquad (3.29)
\end{aligned}$$

Due to

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{|\phi_x\psi_x V_x|}{v^3}\,dx\,d\tau &\leqslant O(1)C_3(\bar{\delta},\overline{M})\int_0^t\big(\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2 + \|\psi_x\|^2\big)\,d\tau \\
&\leqslant O(1)C_3(\bar{\delta},\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big), \qquad (3.30)
\end{aligned}$$

$$\begin{aligned}
\int_0^t\int_{\mathbf{R}}\frac{|\phi_x V_x V_{xx}|}{v^3}\,dx\,d\tau &\leqslant O(1)C_3(\bar{\delta},\overline{M})t_0^{-1}\int_0^t\big(\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2 + \|V_{xx}\|^2\big)\,d\tau \\
&\leqslant O(1)C_3(\bar{\delta},\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big), \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
&\int_0^t\int_{\mathbf{R}}\frac{|\phi_x\phi_{xx} V_x|}{v^3}\,dx\,d\tau \\
&\quad\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}}\frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)\int_0^t\int_{\mathbf{R}}\frac{\phi_x^2 V_x^2}{v^4}\,dx\,d\tau
\end{aligned}$$

$$\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}} \frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)C_3(\bar\delta,\overline{M})\int_0^t\int_{\mathbf{R}} \big(-p'(V+\phi)\big)\phi_x^2\,dx\,d\tau$$

$$\leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}} \frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)C_3(\bar\delta,\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big), \tag{3.32}$$

$$\int_0^t\int_{\mathbf{R}} \frac{|\phi_x^2 V_t|}{v^3}\,dx\,d\tau \leqslant O(1)C_3(\bar\delta,\overline{M})\int_0^t \big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2\,d\tau$$

$$\leqslant O(1)C_3(\bar\delta,\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big), \tag{3.33}$$

we have from (3.30)–(3.33) that

$$I_{12} \leqslant \frac{\varepsilon_1}{5}\int_0^t\int_{\mathbf{R}} \frac{\phi_{xx}^2}{v^2}\,dx\,d\tau + O(1)C_3(\bar\delta,\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big). \tag{3.34}$$

Substituting (3.26)–(3.28) and (3.34) into (3.20), we have

$$\begin{aligned}
&\left\|\frac{\phi_x}{v}\right\|^2 + \int_0^t \left\|\frac{\phi_{xx}}{v}\right\|^2 d\tau \\
&\quad\leqslant O(1)\|\phi_{0x}\|^2 + O(1)C_3(\bar\delta,\overline{M})\big(t_0^{-q_1} + \big\|\big(\phi_0(x),\psi_0(x)\big)\big\|^2\big) \\
&\qquad + O(1)C_3(\bar\delta,\overline{M})\int_0^t \big(t_0^{\gamma}\big\|\sqrt{-p'(V+\phi)}\,\phi_x\big\|^2 + t_0^{-\gamma}\|\phi_{xx}\|^2\big)\left\|\frac{\phi_x}{v}\right\|^2 d\tau.
\end{aligned} \tag{3.35}$$

Having obtained (3.35), (3.19) follows immediately from Gronwall's inequality and Lemma 3.1. This completes the proof of Lemma 3.2. □

We now deduce a uniform lower bound estimate on $v(t,x)$ based on the estimates obtained in Lemmas 3.1 and 3.2. To do so we need to assume that the initial perturbation $(\phi_0(x),\psi_0(x))$ satisfies the following conditions

$$\begin{cases}
\|(\phi_0(x),\psi_0(x))\|^2 \leqslant O(1)t_0^{-q_1}, \\
\|\phi_{0x}(x)\|^2 \leqslant O(1)(1+t_0^{\beta_1}), \\
\|\psi_{0x}(x)\|^2 \leqslant O(1)(1+t_0^{\beta_2}),
\end{cases} \tag{3.36}$$

and q_1, β_1, β_2 satisfy

$$0 < \beta_1 < q_1, \quad \beta_2 \geqslant 0. \tag{3.37}$$

Under the above assumptions, by taking $\gamma = \frac{q_1+\beta_1}{2}$, (3.19) can be rewritten as

$$\left\|\frac{\phi_x}{v}\right\|^2 + \int_0^t \left\|\frac{\phi_{xx}}{v}\right\|^2 d\tau \leqslant O(1)\big(t_0^{\beta_1} + C_3(\bar\delta, \overline{M})t_0^{-q_1}\big)\exp\Big(O(1)C_3(\bar\delta, \overline{M})t_0^{-\frac{q_1-\beta_1}{2}}\Big). \tag{3.38}$$

Moreover, let $\tilde v = \frac{v}{V}$, since

$$\frac{\tilde v_x^2}{\tilde v^2} = \frac{(v_x/V - vV_x/V^2)^2}{v^2/V^2} \leqslant O(1)\left(\frac{\phi_x^2}{v^2} + \frac{V_x^2}{v^2}\right) + O(1)V_x^2,$$

we have from (3.38) and the fact $q_1 < 1$ that

$$\begin{aligned}\left\|\frac{\tilde v_x}{\tilde v}\right\|^2 &\leqslant O(1)\left\|\frac{\phi_x}{v}\right\|^2 + O(1)\left\|\frac{V_x}{v}\right\|^2 + O(1)\|V_x\|^2 \\ &\leqslant O(1)\big(t_0^{\beta_1} + C_3(\bar\delta, \overline{M})t_0^{-q_1}\big)\exp\Big(O(1)C_3(\bar\delta, \overline{M})t_0^{-\frac{q_1-\beta_1}{2}}\Big).\end{aligned} \tag{3.39}$$

Now we use Kanel's method [11] to get a uniform lower bound for $v(t,x)$. To do so, we have from (3.2) and (2.3) that

$$\int_{\mathbf{R}} \frac{(\tilde v - 1)^2}{1+\tilde v}\,dx \leqslant O(1)t_0^{-q_1}. \tag{3.40}$$

Thus if we can choose $t_0 > 0$ sufficiently large such that

$$\begin{cases} C_3(\bar\delta, \overline{M})t_0^{-q_1} < 1, \\ C_3(\bar\delta, \overline{M})t_0^{-\frac{q_1-\beta_1}{2}} < 1, \end{cases} \tag{3.41}$$

then we have from (3.39) that

$$\left\|\frac{\tilde v_x}{\tilde v}\right\|^2 \leqslant O(1)\big(1 + t_0^{\beta_1}\big). \tag{3.42}$$

Hence, if we set

$$\begin{cases} \overline{\Phi}(\tilde v) = \frac{(1-\tilde v)^2}{1+\tilde v} \\ \overline{\Psi}(\tilde v) = \int_1^{\tilde v} \frac{\sqrt{\overline{\Phi}(\eta)}}{\eta}\,d\eta, \end{cases} \tag{3.43}$$

it is easy to see that

$$\overline{\Phi}(\tilde v) \sim \begin{cases} 1, & \text{as } \tilde v \to 0_+, \\ \tilde v, & \text{as } \tilde v \to +\infty, \end{cases} \quad \text{and} \quad \overline{\Psi}(\tilde v) \to \begin{cases} -\infty, & \text{as } \tilde v \to 0_+, \\ +\infty, & \text{as } \tilde v \to +\infty. \end{cases}$$

Since

$$\left|\overline{\Psi}(\tilde{v})\right| = \left| \int_{-\infty}^{x} \frac{\partial \overline{\Psi}(\tilde{v}(t,y))}{\partial y}\, dy \right| \leqslant \left\| \sqrt{\overline{\Phi}(\tilde{v})} \right\| \left\| \frac{\tilde{v}_x}{\tilde{v}} \right\| \leqslant O(1) t_0^{-q_1}\left(1 + t_0^{\beta_1}\right) \leqslant O(1), \tag{3.44}$$

where we have used the fact that $\beta_1 < q_1$.

Since (3.44) implies that $|\overline{\Psi}(\tilde{v})|$ is bounded by a constant independent of t, x, $\bar{\delta}$, t_0 and $\overline{M}$, we can deduce that there exists a positive constant C, which is independent of t, x, $\bar{\delta}$, t_0 and $\overline{M}$, such that

$$C^{-1} \leqslant v(t,x) \leqslant C. \tag{3.45}$$

The above analysis yields the following result:

Lemma 3.3. *Let* $(\phi, \psi)(t,x) \in X_{\bar{\delta},\overline{M}}(0,T)$ *be a solution to the Cauchy problem* (2.2) *defined on the strip* $\Pi(0,T)$ *and assume that we can choose* $t_0 > 0$ *sufficiently large such that* (3.1) *and* (3.41) *hold, then there exists a positive constant* $D > 0$*, which is independent of* t*,* x*,* t_0*,* $\bar{\delta}$*,* $\overline{M}$ *such that*

$$D^{-1} \leqslant v(t,x) \leqslant D. \tag{3.46}$$

Now we turn to prove Theorem 1.1. The main idea is first to deduce certain energy estimates as in Lemma 3.1, and then to use the continuity argument to extend the local solution step by step. Our main trick is to use the parameter t_0 introduced in constructing the smooth approximation of the Riemann solution to control the possible growth of the solution caused by the nonlinearity of the equation and/or by the interactions of waves from different families. It is worth pointing out that to do so we need to ask the initial perturbation to satisfy assumption (1.11).

Since the initial data $(\phi_0(x), \psi_0(x))$ satisfies (1.11) and $v_0(x) \geqslant \delta > 0$, we have from the local existence result Lemma 2.3, especially the estimate (2.6), that there exists a sufficiently small positive constant $t_1 > 0$ such that the Cauchy problem (2.2) admits a unique smooth solution $(\phi(t,x), \psi(t,x))$ on $[0,t_1] \times \mathbf{R}$ and for all $0 \leqslant t \leqslant t_1$, $x \in \mathbf{R}$, $(\phi(t,x), \psi(t,x))$ satisfies

$$\begin{cases} v(t,x) \geqslant \frac{\delta}{2}, \\ \|(\phi(t,x), \psi(t,x))\|^2 \leqslant 2D_1 t_0^{-q_1}, \\ \|\phi_x(t,x)\|^2 \leqslant 2D_2(1 + t_0^{\beta_1}), \\ \|\psi_x(t,x)\|^2 \leqslant 2D_3(1 + t_0^{\beta_2}). \end{cases} \tag{3.47}$$

(3.47) together with the standard Sobolev inequality imply

$$\left\| v(t) \right\|_{L^\infty} \leqslant \left\| V(t,x) \right\|_{L^\infty_{t,x}} + \sqrt{4D_1 D_2}\, t_0^{-\frac{q_1}{2}}\left(1 + t_0^{\frac{\beta_1}{2}}\right), \quad 0 \leqslant t \leqslant t_1. \tag{3.48}$$

Consequently, if we assume that $\beta_1 \leqslant q_1$, then there exists a positive constant $D(D_1, D_2)$ which depend only on D_1, D_2 such that

$$\left\| v(t) \right\|_{L^\infty} \leqslant D(D_1, D_2), \quad 0 \leqslant t \leqslant t_1. \tag{3.49}$$

Having obtained (3.49), one can easily deduce that there exists a sufficiently large constant $\bar{t}_0 > 0$ such that for all $t_0 \geqslant \bar{t}_0$, the solution $(\phi(t,x), \psi(t,x))$ to the Cauchy problem (2.2) obtained above satisfies (3.1) and (3.41) with $\bar{\delta} = \frac{\delta}{2}$, $\overline{M} = D(D_1, D_2)$ for $0 \leqslant t \leqslant t_1$, $x \in \mathbf{R}$. Hence from Lemma 3.3, we can deduce that there exists a positive constant $D_4 > 0$ which is independent of t, x, t_0 such that for $t \in [0, t_1]$, $x \in \mathbf{R}$,

$$D_4^{-1} \leqslant v(t,x) \leqslant D_4. \tag{3.50}$$

Having obtained (3.50), we have from Lemma 3.1 that there exist positive constants D_5, D_6, D_7 such that for all $0 \leqslant t \leqslant t_1$, $x \in \mathbf{R}$

$$\begin{cases} \|(\phi(t,x), \psi(t,x))\|^2 \leqslant D_5 t_0^{-q_1}, \\ \|\phi_x(t,x)\|^2 \leqslant D_6(1 + t_0^{\beta_1}), \\ \|\psi_x(t,x)\|^2 \leqslant D_7(1 + t_0^{\beta_2}) \end{cases} \tag{3.51}$$

provided that $t_0 \geqslant \bar{t}_0$.

Now taking $(\phi(t_1,x), \psi(t_1,x))$ as initial data, we get from (3.50) and (3.51) that

$$\begin{cases} v(t_1,x) \geqslant D_4^{-1}, \\ \|(\phi(t_1,x), \psi(t_1,x))\|^2 \leqslant D_5 t_0^{-q_1}, \\ \|\phi_x(t_1,x)\|^2 \leqslant D_6(1 + t_0^{\beta_1}), \\ \|\psi_x(t_1,x)\|^2 \leqslant D_7(1 + t_0^{\beta_2}), \end{cases} \tag{3.52}$$

and by exploiting the local existence result Lemma 2.3 again, we can deduce that there exists a sufficiently small positive constant $t_2 > 0$ such that the solution $(\phi(t,x), \psi(t,x))$ obtained above can be extended to the time $t = t_1 + t_2$ and for $t_1 \leqslant t \leqslant t_1 + t_2$, $(\phi(t,x), \psi(t,x))$ satisfies

$$\begin{cases} v(t,x) \geqslant \frac{D_4^{-1}}{2}, \\ \|(\phi(t,x), \psi(t,x))\|^2 \leqslant 2D_5 t_0^{-q_1}, \\ \|\phi_x(t,x)\|^2 \leqslant 2D_6(1 + t_0^{\beta_1}), \\ \|\psi_x(t,x)\|^2 \leqslant 2D_7(1 + t_0^{\beta_2}). \end{cases} \tag{3.53}$$

Since without loss of generality, we can assume $D_4^{-1} \leqslant \delta$, $D_1 \leqslant D_5$, $D_2 \leqslant D_6$, $D_3 \leqslant D_7$, consequently from (3.50)–(3.53), we have that the solution $(\phi(t,x), \psi(t,x))$ obtained above satisfies (3.53) for all $0 \leqslant t \leqslant t_1 + t_2$, $x \in \mathbf{R}$.

Notice that (3.53) together with Sobolev's inequality imply

$$\|v(t)\|_{L^\infty([0,t_1+t_2]\times\mathbf{R})} \leqslant \|V(t,x)\|_{L^\infty(\mathbf{R}_+\times\mathbf{R})} + \sqrt{4D_5D_6}\, t_0^{-\frac{q_1}{2}}\left(1 + t_0^{\frac{\beta_1}{2}}\right). \tag{3.54}$$

From (3.54), we know that if $\beta_1 \leqslant q_1$, there exists a positive constant $D(D_5, D_6)$ which depend only on D_5, D_6 such that

$$\|v(t)\|_{L^\infty([0,t_1+t_2]\times\mathbf{R})} \leqslant D(D_5, D_6). \tag{3.55}$$

Having obtained (3.55), we can find an $\bar{\bar{t}}_0 > 0$ which is chosen sufficiently large such that if $t_0 \geqslant \bar{\bar{t}}_0$ and for all $0 \leqslant t \leqslant t_1 + t_2$, the solution $(\phi(t,x), \psi(t,x))$ to the Cauchy problem (2.2) obtained above satisfies (3.1) and (3.41). Consequently from Lemmas 3.1 and 3.3, (3.50) and (3.51) hold true for all $0 \leqslant t \leqslant t_1 + t_2$.

Now taking $(\phi(t_1+t_2,x), \psi(t_1+t_2,x))$ as initial data, since

$$\begin{cases} v(t_1+t_2,x) \geqslant D_4^{-1}, \\ \|(\phi(t_1+t_2,x), \psi(t_1+t_2,x))\|^2 \leqslant D_5 t_0^{-q_1}, \\ \|\phi_x(t_1+t_2,x)\|^2 \leqslant D_6(1+t_0^{\beta_1}), \\ \|\psi_x(t_1+t_2,x)\|^2 \leqslant D_7(1+t_0^{\beta_2}), \end{cases} \tag{3.56}$$

and by exploiting the local existence result Lemma 2.3 once more, we can deduce that the solution $(\phi(t,x), \psi(t,x))$ obtained above can be extended to the time $t = t_1 + 2t_2$ and for $t_1 + t_2 \leqslant t \leqslant t_1 + 2t_2$, $(\phi(t,x), \psi(t,x))$ satisfies (3.53) provided that $t_0 \geqslant \bar{\bar{t}}_0$.

Repeating the above procedure and noticing the constants D_4, D_5 and D_6 are independent of each time step, we can thus extend the solution step by step to the whole $\mathbf{R}_+$ provided that $t_0 \geqslant \max\{\bar{t}_0, \bar{\bar{t}}_0\}$. As a byproduct, one can deduce that if we take $t_0 \geqslant \max\{\bar{t}_0, \bar{\bar{t}}_0\}$, the Cauchy problem (2.2) has a unique global smooth solution $(\phi(t,x), \psi(t,x))$ which satisfies (3.50) and (3.51) for all $t \in \mathbf{R}_+$. This completes the proof of Theorem 1.1.

4. The proof of Theorem 1.2

This section is devoted to prove Theorem 1.2. Before giving the details, we first outline the main idea. Since Lemma 2.4 only gives the lower bound on $v(t,x)$, we perform our energy estimates based on the following a priori assumption: there exists a positive constant $M > 0$ such that for any fixed $T > 0$

$$0 < \delta \leqslant v(t,x) \leqslant M, \quad (t,x) \in [0,T] \times \mathbf{R}. \tag{4.1}$$

In this case, since $v(t,x) \geqslant \delta$ for all $(t,x) \in [0,T] \times \mathbf{R}$, $C_i(\delta, M) = C_i(M)$ $(i = 1,2)$ in (3.1). Based on such an a priori assumption and by choosing the quantity t_0 introduced in the construction of smooth approximation to the rarefaction waves suitably large, we can deduce certain H^1-norm energy estimates (3.2)–(3.4) from which one can deduce that there exist some positive constant $C_3 > 1$ and $C_4 > 1$, which depend only on the system but are independent of $M > 0$ and t_0, such that for all $(t,x) \in [0,T] \times \mathbf{R}$

$$\begin{cases} |u(t,x)| \leqslant C_3(1 + \|(\phi_0, \psi_0)(x)\|_1), \\ 0 < \delta \leqslant v(t,x) \leqslant C_4(1 + \|(\phi_0, \psi_0)(x)\|_1^2) \end{cases} \tag{4.2}$$

provided that we can choose $t_0 > 0$ sufficiently large such that (3.1) holds.

Having obtained (4.2), we can deduce from the continuity argument that if the H^1-norm of the initial perturbation $(\phi_0(x), \psi_0(x))$ satisfies (1.13), we can verify that one can indeed choose t_0 suitably large such that (4.2) holds. Once this is achieved, the desired time asymptotic estimate (1.12) follows immediately. From (3.2)–(3.4) and Lemma 2.2, we get that

$$\int_{\mathbf{R}} \left(\psi^2 + \frac{\phi^2}{v+V} + \psi_x^2 + \phi_x^2 \right)(t)\,dx \leqslant O(1)\big(1 + \|(\phi_0, \psi_0)\|_1^2\big). \tag{4.3}$$

Due to

$$\psi^2(t,x) = 2\int_{-\infty}^{x} \psi(t,y)\psi_x(t,y)\,dy \leqslant \int_{\mathbf{R}} \big(\psi^2 + \psi_x^2\big)(t,y)\,dy,$$

$$\begin{aligned}\frac{\phi^2(t,x)}{v(t,x)+V(t,x)} &= \int_{-\infty}^{x} \left(\frac{2\phi\phi_x}{v+V} - \frac{\phi^2}{(v+V)^2}(\phi_x + 2V_x) \right)(t,x)\,dx \\ &\leqslant O(1)\int_{\mathbf{R}} \left(\frac{\phi^2}{v+V} + \phi_x^2 + V_x^2 \right)(t,x)\,dx,\end{aligned}$$

we can deduce from (4.3) that if we can choose $t_0 > 0$ sufficiently such that (3.1) holds, there exist positive constants $C_3 > 1$ and $C_4 > 1$, which depend only on the system but are independent of t_0 and M, such that (4.2) holds.

Now to complete the proof of (4.2), we only need to show that it is possible to choose t_0 sufficiently large such that (3.1) holds and this is achieved by combining the argument used above to deduce the estimate (4.2) with the continuity argument. It is worth pointing out that it is in this step that we ask the H^1-norm of the initial perturbation to satisfy (1.13).

Note first that if the H^1-norm of the initial perturbation is bounded by a constant independent of t_0, then the argument used above to deduce (4.2) indicates that for each fixed $T > 0$, $u(t,x)$ and $v(t,x)$ are bounded by two constants independent of t_0 and M for $0 \leqslant t \leqslant T$, $x \in \mathbf{R}$ and thus from the standard continuity argument, we can easily choose $t_0 > 0$ sufficiently large such that (3.1) holds and (4.2) is proved.

Now we consider the case when H^1-norm of the initial perturbation depends on $t_0 > 0$, that is, we deal with the case when the H^1-norm of the initial perturbation satisfies (1.13). Define $\kappa(M) = \max\{C_1(M), C_2(M)\}$. In such a case, since $\kappa^{-1}(t_0^\alpha) \to +\infty$ as $t_0 \to +\infty$, we can choose t_0 sufficiently large such that

$$\max\big\{1 + C_0, \max\{v_-, v_m, v_+\}\big\} \leqslant \frac{\sqrt{\kappa^{-1}(t_0^\alpha)}}{8C_1}. \tag{4.4}$$

Consequently from (1.13), we get that

$$\big\|(\phi_0, \psi_0)(x)\big\|_1 \leqslant C_0 + \frac{\sqrt{\kappa^{-1}(t_0^\alpha)}}{8C_1} \leqslant \frac{\sqrt{\kappa^{-1}(t_0^\alpha)}}{4C_1}. \tag{4.5}$$

With (4.5) in hand, we can deduce from Lemma 2.4 and the local existence result Lemma 2.3 that there exists a positive constant $t_1 > 0$ such that the Cauchy problem (2.2) admits a unique smooth solution $(\phi(t,x), \psi(t,x))$ defined on the strip $\Pi(0,t_1)$ and for $0 \leqslant t \leqslant t_1$, $x \in \mathbf{R}$, $(\phi(t,x), \psi(t,x))$ satisfies

$$\begin{cases} v(t,x) \geqslant \delta > 0, \\ \|(\phi(t,x), \psi(t,x))\|_1 \leqslant 2\|(\phi_0(x), \psi_0(x))\|_1. \end{cases} \tag{4.6}$$

From (4.6), we have for $0 \leqslant t \leqslant t_1$ that

$$\begin{aligned} 0 < \delta \leqslant v(t,x) &\leqslant \|V(t,x)\|_{L^\infty_{t,x}} + \|\phi(t,x)\|_1 \leqslant \max\{v_-, v_m, v_+\} + 2\|(\phi_0(x), \psi_0(x))\|_1 \\ &\leqslant \max\{v_-, v_m, v_+\} + \frac{\sqrt{\kappa^{-1}(t^\alpha)}}{2C_1} \leqslant \frac{\sqrt{\kappa^{-1}(t_0^\alpha)}}{C_1} \leqslant \kappa^{-1}(t_0^\alpha). \end{aligned} \tag{4.7}$$

Now for $0 \leqslant t \leqslant t_1$, we use the argument used above to deduce estimate (4.2) with M being replaced by $\kappa^{-1}(t_0^\alpha)$. For such a M, since $0 < \alpha < \frac{1}{8q}$, it is easy to see that we can choose $\bar{t}_0 > 0$ sufficiently large such that (3.1) holds for $t_0 > \bar{t}_0$ and consequently, the results obtained in Lemma 3.1, i.e., (3.2)–(3.4) hold for $0 \leqslant t \leqslant t_1$. That is, (4.3) holds for $0 \leqslant t \leqslant t_1$. Having obtained (4.3), we have by repeating the argument used above to deduce estimate (4.2) that there exists positive constants $C_3 > 1$ and $C_4 > 1$, which are independent of t_0 and M such that (4.2) holds for $(t,x) \in [0, t_1] \times \mathbf{R}$, i.e.,

$$0 < \delta \leqslant v(t,x) \leqslant C_4\left(1 + \|(\phi_0, \psi_0)(x)\|_1^2\right) \leqslant \frac{C_4\kappa^{-1}(t_0^\alpha)}{8C_1^2} \leqslant \kappa^{-1}(t_0^\alpha) \tag{4.8}$$

and

$$|u(t,x)| \leqslant C_3\left(1 + \|(\phi_0, \psi_0)(x)\|_1\right). \tag{4.9}$$

Here it is worth pointing out that in deducing the last inequality in (4.8), we have used assumption (1.13) and the resulting inequality (4.4).

Moreover, as a by-product of (4.8) and (4.3), we can deduce that there exists a positive constant $C_5 > 0$, which is independent of t, x, t_0 and M, such that for $0 \leqslant t \leqslant t_1$

$$\|(\phi(t,x), \psi(t,x))\|_1 \leqslant C_5\|(\phi_0(x), \psi_0(x))\|_1^2. \tag{4.10}$$

Now take $(\phi(t_1,x), \psi(t_1,x))$ as initial data, we have by exploiting the local existence result Lemma 2.3 again that the solution $(\phi(t,x), \psi(t,x))$ to Cauchy problem (2.2) obtained above can be extended to the time step $t = t_1 + t_2$ for some $t_2 > 0$ which depends only on $\|(\phi(t_1,x), \psi(t_1,x))\|_1$ such that for all $(t,x) \in [t_1, t_1 + t_2] \times \mathbf{R}$

$$\begin{cases} v(t,x) \geqslant \delta > 0, \\ \|(\phi(t,x), \psi(t,x))\|_1 \leqslant 2C_5\|(\phi_0(x), \psi_0(x))\|_1^2, \end{cases} \tag{4.11}$$

from which, we have for $t_1 \leqslant t \leqslant t_1 + t_2$ that

$$\begin{aligned} 0 < \delta \leqslant v(t,x) &\leqslant \|V(t,x)\|_{L^\infty_{t,x}} + \|\phi(t,x)\|_1 \leqslant \max\{v_-, v_m, v_+\} + 2C_5\|(\phi_0(x), \psi_0(x))\|_1^2 \\ &\leqslant \max\{v_-, v_m, v_+\} + \frac{C_5\kappa^{-1}(t^\alpha)}{8C_1^2} \leqslant \frac{\kappa^{-1}(t_0^\alpha)}{C_1} \leqslant \kappa^{-1}(t_0^\alpha). \end{aligned} \tag{4.12}$$

(4.12) together with (4.7) imply that for all $0 \leqslant t \leqslant t_1 + t_2$, $0 < \delta \leqslant v(t,x) \leqslant \kappa^{-1}(t_0^\alpha)$. Thus by repeating the argument used above, we can deduce that (4.8)–(4.10) holds for $0 \leqslant t \leqslant t_1 + t_2$ provided that $t_0 > \bar{t}_0$ for some suitably large $\bar{t}_0 > 0$. Repeating the above process and from the continuity argument and noticing that the constants C_3, C_4 and C_5 are independent of each time step, one can easily get (4.2) holds for all $0 \leqslant t < \infty$, $x \in \mathbf{R}$. This completes the proof of Theorem 1.2.

5. The proof of Theorem 1.3

In this section, we prove Theorem 1.3. First, from Lemma 2.4 and the assumption $v_0(x) \geqslant \delta > 0$, we know that if $(v(t,x), u(t,x))$ is a smooth solution to the Cauchy problem (1.1) defined on the strip $\Pi(0,T)$, then we have for each $0 \leqslant t \leqslant T$, $x \in \mathbf{R}$ that

$$v(t,x) \geqslant \delta. \tag{5.1}$$

Moreover, under the additional assumption (1.14), we now deduce certain energy estimates similar to that of (3.2)–(3.4). The main difficulty lies the fact that we cannot hope

$$\left| \frac{|p'(v) - p'(V)|}{\sqrt{-p'(v)}\sqrt{p(v) - p(V) - p'(V)(v-V)}} \right| \leqslant O(1)$$

holds, thus we cannot use the method employed in Lemma 3.1 directly. However, from the proof of Lemma 3.1, we only need to control the following two terms

$$I = \int_0^t \int_{\mathbf{R}} \big(p'(V+\phi) - p'(V)\big) V_x \phi_x \, dx \, d\tau \quad \text{and} \tag{5.2}$$

$$J = \int_0^t \int_{\mathbf{R}} \big(p'(V+\phi) - p'(V)\big) V_x \psi_{xx} \, dx \, d\tau \tag{5.3}$$

suitably.

In fact, by integrations by parts and from the assumption (1.14), I can be estimated as follows

$$\begin{aligned}
I = {} & -\int_0^t \int_{\mathbf{R}} \phi V_{xx} \big(p'(V+\phi) - p'(V)\big) \, dx \, d\tau - \int_0^t \int_{\mathbf{R}} \phi V_x^2 \big(p''(V+\phi) - p''(V)\big) \, dx \, d\tau \\
& - \int_0^t \int_{\mathbf{R}} \phi \phi_x V_x p''(V+\phi) \, dx \, d\tau \\
\leqslant {} & \sup_{v \geqslant \delta} \left(\frac{|v - V|(|p'(v) - p'(V)| + |p''(v) - p''(V)|)}{\Phi(V, v-V)} \right) \int_0^t \big(\|V_{xx}(\tau)\|_{L^\infty} + \|V_x(\tau)\|_{L^\infty}^2 \big) \\
& \times \big\| \sqrt{\Phi(V, v-V)} \big\|^2 \, d\tau
\end{aligned}$$

$$+\sup_{v\geqslant\delta}\left(\frac{|v-V||p''(v)|}{\sqrt{-p'(v)}\sqrt{\Phi(V,v-V)}}\right)\int_0^t \left\|V_x(\tau)\right\|_{L^\infty}\left\|\sqrt{-p'(V+\phi)}\,\phi_x\right\|\left\|\sqrt{\Phi(V,v-V)}\right\|d\tau$$

$$\leqslant\frac{\varepsilon_1}{4}\int_0^t\int_{\mathbf{R}}\bigl(-p'(V+\phi)\bigr)\phi_x^2\,dx\,d\tau+O(1)\int_0^t(1+\tau)^{-1-\frac{1}{2q}}\left\|\sqrt{\Phi(V,v-V)}\right\|^2d\tau. \tag{5.4}$$

Similarly, for J, since $|p'(v)|\leqslant -p'(\delta)$, we have

$$J=-\int_0^t\int_{\mathbf{R}}\psi_x V_{xx}\bigl(p'(V+\phi)-p'(V)\bigr)\,dx\,d\tau-\int_0^t\int_{\mathbf{R}}\psi_x V_x^2\bigl(p''(V+\phi)-p''(V)\bigr)\,dx\,d\tau$$

$$-\int_0^t\int_{\mathbf{R}}\psi_x\phi_x V_x p''(V+\phi)\,dx\,d\tau$$

$$\leqslant\sup_{v\geqslant\delta}\left(\frac{|p'(v)-p'(V)|+|p''(v)-p''(V)|}{\Phi(V,v-V)}\right)^2\int_0^t\bigl(\left\|V_{xx}(\tau)\right\|_{L^\infty}^2+\left\|V_x(\tau)\right\|_{L^\infty}^4\bigr)$$

$$\times\left\|\sqrt{\Phi(V,v-V)}\right\|^2d\tau$$

$$+O(1)\int_0^t\int_{\mathbf{R}}\psi_x^2\,dx\,d\tau+O(1)\sup_{v\geqslant\delta}\left(\frac{|p''(v)|}{|p'(v)|}\right)^2\int_0^t\int_{\mathbf{R}}\bigl|p'(V+\phi)\bigr|^2\phi_x^2\,d\tau$$

$$\leqslant O(1)\int_0^t\int_{\mathbf{R}}\bigl(\psi_x^2-p'(V+\phi)\phi_x^2\bigr)\,dx\,d\tau+O(1)\int_0^t(1+\tau)^{-1-\frac{1}{2q}}\left\|\sqrt{\Phi(V,v-V)}\right\|^2d\tau. \tag{5.5}$$

Having obtained (5.4) and (5.5), we have from the proof of Lemma 3.1 that there exist positive constants C_6, C_7 and C_8 such that the solution $(\phi(t,x),\psi(t,x))$ satisfies

$$\int_{\mathbf{R}}\bigl(\psi^2+\Phi(V,v-V)\bigr)\,dx+\int_0^t\int_{\mathbf{R}}\psi_x^2\,dx\,d\tau+\int_0^t\int_{\mathbf{R}}\bigl(-p'(V+\phi)\phi_x^2\bigr)\,dx\,d\tau$$

$$+\int_0^t\int_{\mathbf{R}}\bigl(p(V+\phi)-p(V)-p'(V)\phi\bigr)V_t\,dx\,d\tau$$

$$\leqslant C_6\bigl(t_0^{-q_1}+\left\|(\psi_0,\phi_0)(x)\right\|^2\bigr), \tag{5.6}$$

$$\int_{\mathbf{R}}\phi_x^2\,dx+\int_0^t\int_{\mathbf{R}}\phi_{xx}^2\,dx\,d\tau\leqslant C_7\bigl(t_0^{-q_1}+\left\|(\psi_0,\phi_0)(x)\right\|^2+\left\|\phi_{0x}(x)\right\|^2\bigr) \tag{5.7}$$

and

$$\int_{\mathbf{R}} \psi_x^2 \, dx + \int_0^t \int_{\mathbf{R}} \psi_{xx}^2 \, dx \, d\tau \leqslant C_8 \big(t_0^{-q_1} + \big\| (\psi_0, \phi_0)(x) \big\|^2 + \big\| \psi_{0x}(x) \big\|^2 \big). \tag{5.8}$$

From (5.6)–(5.8), by repeating the argument to deduce (4.2) from (4.3) used in the proof of Theorem 1.2, we can deduce that there exist positive constants $C_9 > 0$ and $C_{10} > 0$ such that

$$\begin{cases} \delta \leqslant v(t,x) \leqslant C_9, \\ |u(t,x)| \leqslant C_{10}. \end{cases} \tag{5.9}$$

Consequently, (5.6)–(5.8) together with (5.9) imply that there exists a positive constant $C_{11} > 0$ such that for $0 \leqslant t \leqslant T$

$$\big\| \big(\phi(t,x), \psi(t,x) \big) \big\|_1 \leqslant C_{11}. \tag{5.10}$$

Thus by combining the local existence result Lemma 2.3 with the a priori estimate (5.10) obtained above, we can easily deduce that the Cauchy problem (2.2) admits a unique global smooth solution $(\phi(t,x), \psi(t,x))$ satisfying (5.6)–(5.8) for all $t \in \mathbf{R}_+$. This completes the proof of Theorem 1.3.

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